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# Solution for 'Topics in complex analysis'

(17/12/2025)

**H 14.1 (A higher dimensional Schwarz lemma)**
Let $f : B_1(0) \subset \mathbb{C}^n \to \mathbb{C}$ be holomorphic such that $f(0) = 0$. Assume that there exists a constant $M > 0$ such that $|f(z)| \leq M$ for all $z \in B_1(0)$.

a) Show that

$$|f(z)| \leq M\|z\| \qquad \forall z \in B_1(0),$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{C}^n$.

b) If $n = 1$, the equality $|f(z)| = M|z|$ for some $z \neq 0$ implies that $f(z) = Maz$ for some $a \in \partial B_1(0)$, so $f$ is biholomorphic. Show that for $n \geq 2$ there exists a holomorphic function $f : B_1(0) \to B_1(0)$ with $f(0) = 0$ and $\|f(z)\| = \|z\|$ for some $z \in B_1(0) \setminus \{0\}$ that is not even injective.

**Remark**: Cartan's uniqueness theorem states that if $D \subset \mathbb{C}^n$ is a bounded domain and $f : D \to D$ has a fixed point $a \in D$ with $Df(a) = \mathrm{Id}$ then $f(z) = z$ for all $z \in D$.

**Solution H 14.1:**
a) Fix $a \in B_1(0) \setminus \{0\}$ and consider the function $f_a : D_1^1(0) \to D_1^1(0)$ defined by

$$f_a(z) = \frac{f\left(z\frac{a}{\|a\|}\right)}{M}.$$

By the chain rule, $f_a$ is holomorphic and satisfies $f_a(0) = f(0) = 0$. Hence by the one-dimensional Schwarz lemma we deduce that $|f_a(z)| \leq |z|$ for all $z \in D_1^1(0)$. Choosing $z = \|a\|$ we conclude that $|f(a)| \leq M\|a\|$ for all $a \in B_1(0) \setminus \{0\}$. Since $f(0) = 0$, this proves the claim.

b) Consider the function $f(z_1, \ldots, z_n) = (z_1, 0, \ldots, 0)$. Then $f : B_1(0) \to B_1(0)$ is holomorphic and satisfies $f(0) = 0$, but $f$ is not injective when $n \geq 2$. However, we have that $\|f(z_1, 0, \ldots, 0)\| = \|(z_1, 0, \ldots, 0)\|$ for all $|z_1| \leq 1$.

□

**H 14.2 (A stronger version of the identity theorem)**
Let $D \subset \mathbb{C}^n$ be a domain and $f : D \to \mathbb{C}$ be holomorphic. Assume that there exists $a \in D$ such that for every multi-index $\alpha \in (\mathbb{N}_0)^n$ it holds that $D^\alpha f(a) = 0$. Show that $f \equiv 0$.

**Solution H 14.2:**
By Corollary 9.6 (ii) we can write $f$ locally near $a$ as a convergent power series, i.e. there exists $r > 0$ such that for all $z \in B_r(a)$ it holds that

$$f(z) = \sum_{\alpha \in (\mathbb{N}_0)^n} \frac{D^\alpha f(a)}{\alpha!}(z-a)^\alpha.$$

Hence by assumption $f|_{B_r(a)} \equiv 0$, so that by the identity theorem (Corollary 9.4 (ii)) we conclude that $f \equiv 0$.

□

**H 14.3 (Consequences of Hartogs's extension theorem)**
Let $n \geq 2$ and $U \subset \mathbb{C}^n$ be open. Show the following statements.

a) If $f : U \setminus \{a\} \to \mathbb{C}$ is holomorphic, then $f$ can be extended to a holomorphic function $f : U \to \mathbb{C}$.

b) If $K \subset \mathbb{C}^n$ is compact and such that $\mathbb{C}^n \setminus K$ is connected, then every holomorphic function $f : \mathbb{C}^n \setminus K \to \mathbb{C}$ can be extended to an entire function.

c) If $f : U \to \mathbb{C}$ is holomorphic, then $f$ cannot have an isolated zero.

d) If $f : \mathbb{C}^n \to \mathbb{C}$ is entire, then $\{f = 0\}$ is either empty or unbounded.

**Solution H 14.3:**

**Remark**: In order to get an idea of how to apply the version of Hartogs's extension theorem proved in the lecture notes for a) and b), it helps to draw a picture of the case $n = 2$.

a) It is enough to consider the case $a = 0$ and $U = D_R^n(0)$, a polydisc with equal radii $R > 0$. Choose $D = D_R^{n-1}(0)$ and the annulus $A(0, R)$ in Theorem 9.9. Then for any $z' \in D_R^{n-1}(0) \setminus \{0\}$ the function $f$ is holomorphic on a set of the form $B_\varepsilon(z') \times B_R(0)$, for $\varepsilon > 0$ sufficiently small depending on $z'$. Hence we can extend it to a holomorphic function on $D_R^n(0)$, which proves the claim.

b) Since $K \subset \mathbb{C}^n$ is compact, there exists $R > 0$ such that $K \subset \overline{D_R^n(0)}$. We will prove that $f|_{\mathbb{C}^{n-1} \times \mathbb{C} \setminus \overline{D_R^1(0)}}$ can be extended to an entire function. By the identity theorem it then follows that this extension also extends $f : \mathbb{C}^n \setminus K \to \mathbb{C}$ (here we use the assumption that $\mathbb{C}^n \setminus K$ is connected). We apply Theorem 9.9 with $D = \mathbb{C}^{n-1}$ and the annulus $A(R, \infty) := \mathbb{C} \setminus \overline{D_R^1(0)}$. Note that $f$ is holomorphic on $D \times A(R, \infty)$. Moreover, for any $z' \in \mathbb{C}^{n-1} \setminus \overline{D_R^{n-1}(0)}$ the function $f$ is holomorphic on a neighborhood of $\{z'\} \times \mathbb{C}$, so that by Theorem 9.9 it can be extended to a holomorphic function on $\mathbb{C}^{n-1} \times \mathbb{C} = \mathbb{C}^n$. This proves the claim.

c) If $f$ has an isolated zero at $a \in U$, then there exists $r > 0$ such that $1/f : B_r(a) \setminus \{a\} \to \mathbb{C}$ is holomorphic. However, $1/f$ is unbounded in a neighborhood of $a$, so it cannot be extended to a holomorphic function on $B_r(a)$. This contradicts a).

d) Assume that $\{f = 0\}$ is non-empty. If it were compact, then there would exist $R > 0$ such that the map $1/f : \mathbb{C} \setminus \overline{B_R(0)} \to \mathbb{C}$ is holomorphic. By b) this map can be extended to an entire function, which we call $g$. Then $f \cdot g = 1$ on an open set. Hence $f \cdot g \equiv 1$ by the identity theorem. This yields a contradiction to the fact that $f$ has a zero.

□